Scattering in chiral strong backgrounds

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With L. Mason & A. Sharma [2003.13501,2010.14996, wip]
see also work with E. Casali, A. Ilderton & S. Nekovar

Amplitudes: what's it all about?

To compute S-matrix, usually follow recipe:

- Perturbation theory around trivial background
- Space-time Lagrangian $\rightarrow$ Feynman rules
- Draw diagrams & compute

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- Lots of progress, especially for massless QFTs
- *All-multiplicity* formulae at tree-level and beyond
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New formulation(s) of perturbative QFT?

Examples

Consider Yang-Mills theory. At tree-level, we know *everything*:

$$A_{n,0}^{(0)} = \delta^4\left(\sum_{i=1}^{n} k_i\right) \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \qquad \texttt{[Parke-Taylor]}$$

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$$A^{(0)}_{n,k} = \int \frac{\prod_{r=0}^{k+1} \mathrm{d}^{4|4} U_r}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})} \prod_{i=1}^{n} \frac{\mathrm{d}\sigma_i\, \mathcal{A}_i(Z(\sigma_i))}{\sigma_i - \sigma_{i+1}}$$ [Roiban-Spradlin-Volovich-Witten]

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[Roiban-Spradlin-Volovich-Witten]

$$A_{n}^{(0)} = \delta^d\left(\sum_{i=1}^{n} k_i\right) \int \mathrm{d}\mu_n \prod_{j=1}^{n} \delta(\mathcal{S}_j) \prod_{i=1}^{n} \frac{1}{\sigma_i - \sigma_{i+1}}\, \mathrm{Pf}'(M)$$

[Cachazo-He-Yuan]

Strong backgrounds

What about amplitudes on *strong* (non-trivial) backgrounds?

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MANY reasons to be interested:

- Strong field QED/laser physics – (electromagnetic plane waves)
- High-energy hadron scattering/colour glass condensates – (Yang-Mills plane waves & shockwaves)
- Gravitational waves – (gravitational plane waves & shockwaves)
- Cosmology – (de Sitter or FLRW space-times)
- Strongly-coupled CFTs/holography – (anti-de Sitter)
- Non-perturbative physics in general

Knowledge gap

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Best results at tree-level: *4-points*

- QED in plane wave [Ilderton,...]
- YM/GR in AdS [d'Hoker-Freedman-et al., Raju]
- Contact amplitudes in AdS [Nagaraj-Ponomarev]
- YM in plane wave [TA-Casali-Mason-Nekovar]

But a novel formulation of QFT should work on *any* background...

Today

Question:

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Strategy:

Consider scattering on strong *chiral*, *source-free*, *asymptotically flat* backgrounds in 4d

Projective geometry of $\mathscr{I}$

Null boundary of $\mathbb{R}^{1,3}$: $\mathscr{I} = \mathscr{I}^- \cup \mathscr{I}^+$, $\mathscr{I}^\pm \cong \mathbb{R} \times S^2$.
Homogeneous coords on $\mathscr{I}^+$ [Sparling, Eastwood-Tod] :

$$(u,\, \lambda_\alpha,\, \bar{\lambda}_{\dot\alpha}) \sim \left(|a|^2 u,\, a\lambda_\alpha,\, \bar{a}\bar{\lambda}_{\dot\alpha}\right), \qquad \forall a \in \mathbb{C}^*$$

$\mathrm{d}s^2_{\mathscr{I}^+} = 0 \times \mathrm{d}u^2 + \mathrm{D}\lambda\, \mathrm{D}\bar{\lambda}$, where $\mathrm{D}\lambda := \langle \lambda\, \mathrm{d}\lambda \rangle = \lambda^\alpha \mathrm{d}\lambda_\alpha$

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Line bundles $\mathcal{O}(p,q) \to \mathscr{I}^+$ encode spin/conformal weights

$$s = \frac{p-q}{2}\,, \qquad w = \frac{p+q}{2}$$

Radiative gauge fields

Asymptotically flat gauge field:

$$A|_{\mathscr{I}^+} = \mathcal{A}^0(u, \lambda, \bar{\lambda})\, \mathrm{D}\lambda + \bar{\mathcal{A}}^0(u, \lambda, \bar{\lambda})\, \mathrm{D}\bar{\lambda}$$

$\mathcal{A}^0$ in $\mathcal{O}(-2, 0) \otimes \mathfrak{g}$, $\bar{\mathcal{A}}^0$ in $\mathcal{O}(0, -2) \otimes \mathfrak{g}$

Define *broadcasting function* $\phi_2 = \partial_u \mathcal{A}^0$ in $\mathcal{O}(-3, -1) \otimes \mathfrak{g}$.

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Radiative gauge field: uniquely characterized by ϕ_2, $\bar{\phi}_2$.

Key example: Yang-Mills plane waves [Trautman, Basler-Hadicke, TA-Casali-Mason-Nekovar]

Self-dual radiative fields

Field strength decomposes into SD/ASD parts:

$$F = F^{+} + F^{-}\,, \qquad *F^{\pm} = \pm \mathrm{i}\, F^{\pm}$$

Asymptotically

$$F^{+}|_{\mathscr{I}^{+}} = \partial_u \bar{\mathcal{A}}^{0}\, \mathrm{d}u \wedge \mathrm{D}\bar{\lambda}\,, \qquad F^{-}|_{\mathscr{I}^{+}} = \partial_u \mathcal{A}^{0}\, \mathrm{d}u \wedge \mathrm{D}\lambda$$

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Def: A *SD radiative gauge field* is a rad. gauge field with $\mathcal{A}^0 = 0$, $\tilde{\mathcal{A}}^0 \neq 0$.

Summary

SD radiative gauge fields:

- complex/chiral 4d gauge fields
- purely radiative, source free
- characterized by free data $\tilde{\phi}_2 = \partial_u \tilde{\mathcal{A}}^0$
- asymptotically flat $\rightarrow$ S-matrix natural observable
- include (multi-)plane waves

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Similar story for gravity (SD radiative space-times)

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...any many new subtleties vs. trivial background:

- No momentum conservation – integrals always left over due to $\tilde{\mathcal{A}}^0$
- Memory effect [Bieri-Garfinkle, Pasterski, TA-Casali-Mason-Nekovar]
- Tails [Günther-Wünsch, Mason, Harte]

Seems like a hard problem. However...

Shift YM action by topological term:

$$-\frac{1}{2\,\mathrm{g}^2}\int \mathrm{tr}\, F\wedge *F + \frac{1}{8\,\mathrm{g}^2}\int \mathrm{tr}\, F\wedge F = -\frac{1}{2\,\mathrm{g}^2}\int \mathrm{tr}\, F^- \wedge F^-$$

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Introduce Lagrange multiplier $B\in\Omega^2_-(\mathfrak{g})$.

$-\frac{1}{2\,\mathrm{g}^2}\int \mathrm{tr}\, F^-\wedge F^-$ equivalent to

$$S[A,B] = \int \mathrm{tr}\, F^-\wedge B + \frac{\mathrm{g}^2}{2}\int \mathrm{tr}\, B\wedge B$$

Field equations:

$$\boxed{F^- = -\mathrm{g}^2\, B\,, \qquad \mathrm{D}B = 0}$$

Upshot

Yang-Mills admits a pert. expansion around SD sector

[Chalmers-Siegel]

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Need: something that manifests the integrability/triviality of the SD background...

Twistor theory

Twistor space: $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ homog. coords. on $\mathbb{CP}^3$

$$\mathbb{PT} = \mathbb{CP}^3 \setminus \{\lambda_\alpha = 0\}$$

$x \in \mathbb{C}^4$ given by $X \cong \mathbb{CP}^1 \subset \mathbb{PT}$ via $\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha$

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On a flat background:

- Massless free fields $\leftrightarrow$ cohomology on $\mathbb{PT}$ [Penrose, Sparling, Eastwood-Penrose-Wells]
- Representation for on-shell scattering kinematics [Hodges]
- Full tree-level S-matrix of $\mathcal{N} = 4$ SYM [Witten, Berkovits, Roiban-Spradlin-Volovich]
- Full tree-level S-matrix of $\mathcal{N} = 8$ SUGRA [Cachazo-Skinner]

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Theorem [Ward, 1977]

There is a 1:1 correspondence between:

- SD SU(N) Yang-Mills fields on $\mathbb{C}^4$, and
- rank N holomorphic vector bundles $E \rightarrow \mathbb{PT}$ trivial on every $X \subset \mathbb{PT}$ (+ technical conditions)

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Punchline: twistor theory *trivializes* the SD sector

Upshot

SD sector encoded by integrable partial connection on $E \to \mathbb{PT}$:

$$\bar{D} = \bar{\partial} + \mathsf{A}\,, \qquad \mathsf{A} \in \Omega^{0,1}(\mathrm{End}E)\,, \qquad \bar{D}^2 = 0$$

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For SD radiative fields [Sparling, Newman, Eastwood-Tod]

$$\boxed{\mathsf{A}(Z) = \tilde{\mathcal{A}}^0(\mu^{\dot\alpha}\bar\lambda_{\dot\alpha}, \lambda, \bar\lambda)\, \mathrm{D}\bar\lambda}$$

Technical assumptions $\Rightarrow E|_X$ holomorphically trivial:

$$\exists\, \mathsf{H}(x,\lambda,\bar{\lambda}) \;:\quad \bar{D}|_X \mathsf{H} = 0$$

[Sparling, Mason-Skinner]

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[Sparling, Mason-Skinner]

H encodes SD rad. gauge field on $\mathbb{R}^{1,3}$:

$$\mathsf{H}^{-1}\,\lambda^{\alpha}\partial_{\alpha\dot{\alpha}}\mathsf{H} = -\mathrm{i}\,\lambda^{\alpha}\,A_{\alpha\dot{\alpha}}(x)$$

Example: Cartan-valued background

Suppose $\tilde{\mathcal{A}}^0$ valued in $\mathfrak{h} \subset \mathfrak{g}$. Then:

$$\mathsf{H}(x, \lambda) = \exp\left[-g(x, \lambda)\right]$$

where $\mathsf{A}|_x = \bar{\partial}|_X g \Rightarrow$

$$g(x, \lambda) = \frac{1}{2\,\pi\,\mathrm{i}} \int_X \frac{\mathrm{D}\lambda' \wedge \mathrm{D}\bar{\lambda}'}{\langle \lambda\, \lambda' \rangle}\, \frac{\langle o\, \lambda \rangle}{\langle o\, \lambda' \rangle}\, \tilde{\mathcal{A}}^0(x, \lambda')$$

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Recover SD rad. field by Kirchoff-d'Adhemar formula:

$$A_{\alpha\dot{\alpha}}(x) = \frac{o_\alpha}{2\,\pi} \int_X \frac{\mathrm{D}\lambda \wedge \mathrm{D}\bar{\lambda}}{\langle o\,\lambda \rangle}\, \bar{\lambda}_{\dot{\alpha}}\, \tilde{\phi}_2(x,\lambda)$$

Gluon perturbations

Gluon perturbations encoded by cohomology:
$+$ helicity gluon $\leftrightarrow a \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O} \otimes \mathrm{End} E)$
$-$ helicity gluon $\leftrightarrow b \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathrm{End} E)$

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For gluons with asymp. momentum $k_{\alpha\dot{\alpha}} = \kappa_\alpha \tilde{\kappa}_{\dot{\alpha}}$

$$a = \mathsf{T}\, \frac{\langle \xi\, \lambda\rangle}{\langle \xi\, \kappa\rangle} \bar{\partial} \left(\frac{1}{\langle \lambda\, \kappa\rangle} \right) \mathrm{e}^{\mathrm{i} \frac{\langle \xi\, \kappa\rangle}{\langle \xi\, \lambda\rangle} [\mu\, \tilde{\kappa}]}$$

$$b = \mathsf{T}\, \frac{\langle \xi\, \kappa\rangle^3}{\langle \xi\, \lambda\rangle^3} \bar{\partial} \left(\frac{1}{\langle \lambda\, \kappa\rangle} \right) \mathrm{e}^{\mathrm{i} \frac{\langle \xi\, \kappa\rangle}{\langle \xi\, \lambda\rangle} [\mu\, \tilde{\kappa}]}$$

MHV amplitudes

Generating functional:

$$\frac{1}{\mathrm{g}^2}\int \mathrm{d}^4x\,\mathrm{tr}\left(B^{\alpha\beta}\,B_{\alpha\beta}\right)\,, \qquad \text{where} \quad D^{\alpha\dot{\alpha}}B_{\alpha\beta}=0$$

for $D_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} - \mathrm{i}A_{\alpha\dot{\alpha}}$, a SD gauge connection

$+$-helicity gluons $\leftrightarrow$ expand SD background [Mason-Skinner]

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Key idea: take SD rad. background $\oplus$ $+$ helicity gluons

Difficult on space-time, easy on twistor space!

MHV generating functional 2.0

Generating functional in $\mathbb{PT}$:

$$\int \mathrm{d}^4 x \int_{X_1 \times X_2} \mathrm{D}\lambda_1 \, \mathrm{D}\lambda_2 \, \langle \lambda_1 \, \lambda_2 \rangle^2 \, \mathrm{tr} \left[\widehat{\mathsf{H}}_1^{-1} \, b_1 \, \widehat{\mathsf{H}}_1 \, \widehat{\mathsf{H}}_2^{-1} \, b_2 \, \widehat{\mathsf{H}}_2 \right]$$

- $b_{1,2} \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathrm{End} E)$
- $\mathsf{H}_{1,2} = \mathsf{H}(x, \lambda_{1,2})$ holomorphic frames for

$$\bar{\partial} + \mathsf{A} + \sum_{i=3}^{n} a_i \, , \qquad a_i \in H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O} \otimes \mathrm{End} E)$$

Perturbative expansion

Expand $\widehat{\mathrm{H}}_1\,\widehat{\mathrm{H}}_2^{-1}$ as Born series in $\{a_i\}$:

$$\widehat{\mathrm{H}}_1\,\widehat{\mathrm{H}}_2^{-1} = \sum_{m=0}^{\infty}\left(\frac{-1}{2\pi\mathrm{i}}\right)^m \int_{X^m} \frac{\mathrm{H}_1\,\langle 1\,2\rangle}{\langle m+2\,2\rangle}\left(\prod_{p=3}^{m+2} \frac{\mathrm{H}_p^{-1}\,a_p\,\mathrm{H}_p\,\mathrm{D}\lambda_p}{\langle p-1\,p\rangle}\right)\mathrm{H}_2^{-1}$$

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Final result (trivially extended to $\mathcal{N}=4$ SUSY):

$$\boxed{\int \mathrm{d}^{4|8}x \int_{X^n} \mathrm{tr}\left(\prod_{i=1}^{n}\frac{\mathrm{D}\lambda_i\,\mathrm{H}_i^{-1}\,a_i\,\mathrm{H}_i}{\langle i\,i+1\rangle}\right)}$$

Example: Cartan-valued background

$\mathbb{CP}^1$ integrals can be performed explicitly

Let gluons r, s have negative helicity:

$$\frac{\langle \kappa_r \, \kappa_s \rangle^4}{\langle \kappa_1 \, \kappa_2 \rangle \cdots \langle \kappa_n \, \kappa_1 \rangle} \int \mathrm{d}^4 x \, \exp\left[\sum_{i=1}^{n} (\mathrm{i}\, k_i \cdot x + e_i \, g(x, \kappa_i)) \right]$$

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Further simplification for SD plane wave backgrounds

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Other sanity checks & features:

- Explicit checks at 3- and 4-points
- Perturbative limit ($\text{MHV}_n +$ background $\rightarrow \text{MHV}_{n+1}$)
- Flat background limit

Full tree-level S-matrix?

Easy guess for N^{k}MHV, based on holomorphic maps
$Z : \Sigma \cong \mathbb{CP}^1 \to \mathbb{PT}$

$$\int \frac{\prod_{r=0}^{k+1} \mathrm{d}^{4|4} U_r}{\mathrm{vol}\, \mathrm{GL}(2,\mathbb{C})} \, \mathrm{tr} \left(\prod_{i=1}^{n} \frac{\mathrm{d}\sigma_i \, \mathsf{H}_i^{-1} \, a_i(Z(\sigma_i)) \, \mathsf{H}_i}{\sigma_i - \sigma_{i+1}} \right)$$

where:

- $Z(\sigma) = \sum_{r=0}^{k+1} U_r \, \sigma^r$ is a degree $k+1$ holomorphic map
- $\{\sigma_i\} \subset \Sigma$ punctures on $\Sigma \cong \mathbb{CP}^1$
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Only $4(k+1)$ integrals for N^{k}MHV!

Currently just a conjecture...but passes many tests!

Summary

Upshot: it *is* possible to make all-multiplicity statements in strong backgrounds!

Also a (more complicated) version of this story for **gravity**!

Many exciting things to do:

- Prove N^kMHV formula for $k \geq 1$
- Possible pheno applications (backreaction, beam depletion)
- Double copy on SD rad. backgrounds
- Generalize to non-chiral rad. backgrounds
- Other SD backgrounds (dyons, instantons)

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Thanks!

Gravity formula

MHV graviton amplitude on SD rad. space-time:

$$\int_{\overline{\mathcal{M}}_{n,0}(\mathbb{PT},d)} \mathrm{d}\mu_d \, \det{}'(\mathbb{H}^{\vee}) \sum_{t=0}^{n-d-3} \sum_{p_1,\ldots,p_t} \det{}'(\mathbb{H}[\mathbf{a}]) \, \mathrm{e}^{\mathrm{i}\, F_n} \prod_{\mathrm{m}=1}^{t} \mathcal{N}^{(p_{\mathrm{m}}-2)} \, .$$